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EIGEN THEORY OF VISCOELASTIC MECHANICS FOR ANISOTROPIC SOLIDS

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ABSTRACT Anisotropic viscoelastic mechanics is studied under anisotropic subspace. It is proved that there also exist the eigen properties for viscoelastic medium. The modal Maxwell's equation, modal dynamical equation (or modal equilibrium equation) and modal compatibility equation are obtained. Based on them, a new theory of anisotropic viscoelastic mechanics is presented. The advantages of the theory are as follows: 1) the equations are all scalar, and independent of each other. The number of equations is equal to that of anisotropic subspaces, 2) no matter how complicated the anisotropy of solids may be, the form of the definite equation and the boundary condition are in common and explicit, 3) there is no distinction between the force method and the displacement method for statics, that is, the equilibrium equation and the compatibility equation are indistinguishable under the mechanical space, 4) each modal equation has a definite physical meaning, for example, the modal equations of order one and order two express the volume change and shear deformation respectively for isotropic solids, 5) there also exist the potential functions which are similar to the stress functions of elastic mechanics for viscoelastic mechanics, but they are not man-made, 6) the final solution of stress or strain is given in the form of modal superimposition, which is suitable to the proximate calculation in engineering.

KEY WORDS anisotropy, viscoelasticity, eigen theory, modal equation

I. INTRODUCTION

Anisotropy and viscoelasticity are two most important mechanical properties in many engineering materials, such as rock and soil, and also two difficult theoretical problems. It is well known that classical viscoelastic mechanics is composed of the dynamical equation and the compatibility equation of continuum and the anisotropic viscoelastic constitutive equation based on Boltzmann's superimposition principle. They are all tensors because the equations are all established under geometrical space, which results in great difficulty in solving. Although big progress[1,2] has been made in the anisotropic viscoelastic mechanics over the years, the system of the classical viscoelastic mechanics has not changed since the research is confined to constitutive equations. The eigen theory given here describes the anisotropic viscoelastic mechanics under mechanical space, that is, the anisotropic subspace of solids. Because there exists spectrum decomposability in both the elastic coefficients matrix and the viscous coefficients matrix, the equations of the classical viscoelastic mechanics can be projected on the mechanical space. Then, the scalar and normal mechanical equations can be obtained. They are independent of each other and can be solved as one-dimensional viscoelastic equation. The final solution can be obtained by

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modal superimposition.

The eigen theory presented by the author is intended to deal with problems in anisotropic mechanics. The author was enlightened by the concepts of eigen elasticity[3-6]. While finishing studies on the eigen theory of the anisotropic elastic mechanics and its operationalized principle[7-10], the author gave the eigen expression for the constitutive equations of anisotropic viscoelasticity and anisotropic nonlinearity[11,12], and also for the failure theory and strength theory of anisotropic solids[13,14], laying a foundation for this paper.

II. EIGEN EXPRESSION OF ANISOTROPIC VISCOELASTIC CONSTITUTIVE EQUATION

Supposing that the viscoelastic deformation of a solid is composed of an elastic and a viscous part. They obey the generalized Hooke's law of elasticity and the generalized Newton's law of viscosity, respectively

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}^{e}, \qquad \boldsymbol{\sigma} = \boldsymbol{D}\dot{\boldsymbol{\varepsilon}}^{d} \tag{1,2}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are the elastic coefficients matrix and the viscous coefficient matrix of solid respectively. So, the total viscoelastic deformation rate can be obtained from Eqs. (1) and (2).

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{e} + \dot{\boldsymbol{\varepsilon}}^{d} = (\boldsymbol{C}^{-1}\nabla_t + \boldsymbol{D}^{-1})\boldsymbol{\sigma} \tag{3}$$

where $\nabla_t = \partial/\partial t$.

According to the eigen concepts of elastic solids[3-6], there exists the following elastic eigen equation

$$\left(\boldsymbol{C}^{-1} - \frac{1}{\lambda}\boldsymbol{I}\right)\boldsymbol{\varphi} = 0 \tag{4}$$

Equation (4) has six real eigenvalues $\lambda_i (i = 1, 2, \cdots, 6)$ and six corresponding eigenvectors. The former is eigen elasticity (Kelvin modulus) and is not related to coordinates while the latter is mechanical space and indicates the anisotropic direction of solid. Thus, the flexible coefficients matrix can be decomposed spectrally

$$\boldsymbol{C}^{-1} = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{\mathrm{T}} \tag{5}$$

where $\boldsymbol{\Phi}$ is eigen modal matrix and is orthogonal and symmetric. $\boldsymbol{\Lambda}$ is eigen flexible matrix and is diagonal.

It is proved[11] that the eigenvector of dissipation deformation of solid is consistent with that of elastic deformation under the condition close to equilibrium. Thus, there also exists the eigen equation similar to Eq. (4) as follows:

$$\left(\boldsymbol{D}^{-1} - \frac{1}{d}\boldsymbol{I}\right)\boldsymbol{\varphi} = 0 \tag{6}$$

Equation (6) also has six real eigenvalues $d_i (i = 1, 2, \cdots, 6)$ that represent eigen viscosity and are in same anisotropic subspace with eigen elasticity of the same order. Thus, the viscous coefficients matrix can also be decomposed spectrally under same mechanical space

$$\boldsymbol{D}^{-1} = \boldsymbol{\Phi}\boldsymbol{\Gamma}\boldsymbol{\Phi}^{\mathrm{T}} \tag{7}$$

where $\boldsymbol{\Gamma}$ is the reversal of eigen viscous matrix, and is also diagonal.

Substituting Eqs. (5) and (7) into Eq. (3), we get

$$\nabla_t \boldsymbol{\varepsilon} = \boldsymbol{\Phi}(\boldsymbol{\Lambda}\nabla_t + \boldsymbol{\Gamma})\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\sigma} \tag{8}$$

Defining modal stress and modal strain as following[6]

$$\boldsymbol{\sigma}^{*} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\sigma} \tag{9}$$

$$\boldsymbol{\varepsilon}^{*} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varepsilon} \tag{10}$$

where $\boldsymbol{\sigma}^*$ and $\boldsymbol{\varepsilon}^*$ are the stress and strain observed under the mechanical space. Equations (9) and (10) are called the representation conversion relationship.

Substituting Eqs. (9) and (10) into Eq. (8), the differential constitutive equation of anisotropic viscoelasticity under the mechanical space is obtained as follows:

$$(\boldsymbol{\Lambda}\nabla_t + \boldsymbol{\Gamma})\boldsymbol{\sigma}^* = \nabla_t\boldsymbol{\varepsilon}^* \tag{11}$$

Rewriting Eq. (11) in the form of scalar, we have

$$\left(\frac{1}{\lambda_i}\nabla_t + \frac{1}{d_i}\right)\sigma_i^* = \nabla_t\varepsilon_i^*, \qquad i = 1,2,\cdots,6 \tag{12}$$

They are six independent, differential and scalar equations for general anisotropic solids, and have the same form with Maxwell's.

III. EIGEN DYNAMICAL EQUATION OF ANISOTROPIC VISCOELASTIC SOLIDS

Neglecting the body force, the classical dynamical equation of continuum under the geometrical space can be written in the form of matrix[8]

$$\boldsymbol{\Delta\sigma} = \rho\nabla_t\boldsymbol{\varepsilon} \tag{13}$$

in which $\boldsymbol{\Delta}$ is a symmetrical differential operator matrix

$$\boldsymbol{\Delta} = \begin{bmatrix} \partial_{11} & 0 & 0 & 0 & \partial_{31} & \partial_{21} \\ & \partial_{22} & 0 & \partial_{32} & 0 & \partial_{21} \\ & & \partial_{33} & \partial_{32} & \partial_{31} & 0 \\ & & & (\partial_{33} + \partial_{22}) & \partial_{21} & \partial_{31} \\ & \text{sym.} & & & (\partial_{33} + \partial_{11}) & \partial_{32} \\ & & & & & (\partial_{22} + \partial_{11}) \end{bmatrix} \tag{14}$$

where $\partial_{ij} = \partial_{ji} = \partial/\partial x_i \partial x_j$, and $\nabla_{tt} = \partial^2/\partial t \partial t$.

Let $\boldsymbol{\varepsilon} = \beta\alpha\boldsymbol{\varphi}$, where β is an unknown time operator, α an arbitrary time-space variable, and $\boldsymbol{\varphi}$ an unknown vector. According to Eq. (3), if $\boldsymbol{\sigma} = \alpha\boldsymbol{\varphi}$, β and $\boldsymbol{\varphi}$ must be nonzero solution of the following eigen equation

$$[(\boldsymbol{C}^{-1}\nabla_t + \boldsymbol{D}^{-1}) - \beta\boldsymbol{I}]\boldsymbol{\varphi} = \boldsymbol{0} \tag{15}$$

It is seen from above equation that β and $\boldsymbol{\varphi}$ are eigenvalue and eigenvector of the viscoelastic operator matrix respectively. The latter is just the anisotropic subspace, and the former is as follows by means of Eqs. (12)

$$\beta_i = \frac{1}{\lambda_i}\nabla_t + \frac{1}{d_i}, \qquad i = 1,2,\cdots,6 \tag{16}$$

Thus, the viscoelastic operator matrix can also be decomposed spectrally

$$\boldsymbol{C}^{-1}\nabla_t - \boldsymbol{D}^{-1} = \boldsymbol{\Phi\Pi\Phi}^{\mathrm{T}} \tag{17}$$

where $\boldsymbol{\Pi}$ is eigen viscoelastic operator matrix, and is diagonal. Its elements are given in Eqs. (16).

Substituting the strain rate vector $\boldsymbol{\varepsilon} = \beta\alpha\boldsymbol{\varphi}$ and stress vector $\boldsymbol{\sigma} = \alpha\boldsymbol{\varphi}$ into Eq. (13) respectively, we have

$$\boldsymbol{\Delta}(\alpha\boldsymbol{\varphi}) = \rho\nabla_t\beta(\alpha\boldsymbol{\varphi}) \tag{18}$$

It is seen from Eq. (18) that under the condition of viscoelasticity, the geometrical differential operator matrix in the dynamical equation also has the eigen property, and the following eigen equation holds

$$(\boldsymbol{\Delta} - \eta\boldsymbol{I})(\alpha\boldsymbol{\varphi}) = \boldsymbol{0} \tag{19}$$

Transposing Eq. (19), we have

$$\alpha\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{\Delta}-\eta\boldsymbol{I})=\boldsymbol{0} \tag{20}$$

in which α can not be zero, otherwise, there will be zero response. So, we have

$$\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{\Delta}-\eta\boldsymbol{I})=\boldsymbol{0} \tag{21}$$

It is seen from Eq. (21) that $\eta=\rho\nabla_t\beta$ and $\boldsymbol{\varphi}$ are also the eigenvalue and eigenvector of the geometrical differential operator matrix respectively. Because $\boldsymbol{\varphi}$ is the basic vector of anisotropic subspace of viscoelastic solids, if we project the geometrical differential operator of the dynamical equation on the mechanical space, its eigenvalue will be proportional to the time differential operator. So, the geometrical differential operator matrix can also be decomposed spectrally

$$\boldsymbol{\Delta}=\boldsymbol{\Phi}\boldsymbol{\Gamma}\boldsymbol{\Phi}^{\mathrm{T}} \tag{22}$$

where $\boldsymbol{\Gamma}$ is eigen geometrical differential operator matrix, and is diagonal.

Substituting Eq. (22) into Eq. (13), and using Eqs. (9) and (10), we obtain

$$\boldsymbol{\Gamma}\boldsymbol{\sigma}^{*}=\rho\nabla_t\boldsymbol{\varepsilon}^{*} \tag{23}$$

Rewriting Eq. (23) in the form of scalar, it becomes

$$\eta_i\sigma_i^{*}=\rho\nabla_t\dot{\varepsilon}_i^{*}, \qquad i=1,2,\cdots,6 \tag{24}$$

Using Eqs. (12), above equation becomes

$$\eta_i\sigma_i^{*}=\rho\nabla_t\left(\frac{1}{\lambda_i}\nabla_t+\frac{1}{d_i}\right)\sigma_i^{*}, \qquad i=1,2,\cdots,6 \tag{25}$$

They are the eigen dynamical equations of viscoelastic solids, with six independent differential and scalar equations. When the viscous coefficients $d_i \to \infty$ ($i=1,2,\cdots,6$), they will go back to the dynamical equations of elastic solids[7].

Ⅳ. EIGEN STATIC EQUATION OF ANISOTROPIC VISCOELASTIC SOLIDS

Neglecting the inertial force, Eqs. (25) becomes the equilibrium one. Rewriting the eigen differential operator η_i as Δ_i^{*}, we have

$$\Delta_i^{*}\dot{\sigma}_i^{*}=0, \qquad i=1,2,\cdots,6 \tag{26}$$

where Δ_i^{*} is called as stress operator.

It is proved[8] that the compatibility equation of continuum can also be written in the eigen form under the mechanical space

$$\nabla_i^{*}\dot{\varepsilon}_i^{*}=0, \qquad i=1,2,\cdots,6 \tag{27}$$

where ∇_i^{*} is called strain operator and is expressed as

$$\nabla_i^{*}=\boldsymbol{\varphi}_i^{\mathrm{T}}\nabla\boldsymbol{\varphi}_i, \qquad i=1,2,\cdots,6 \tag{28}$$

in which ∇ is also a symmetrical differential operator matrix

$$\nabla=\begin{bmatrix} 0 & \partial_{33} & \partial_{22} & -\partial_{23} & 0 & 0 \\ & 0 & \partial_{11} & 0 & -\partial_{13} & 0 \\ & & 0 & 0 & 0 & -\partial_{12} \\ & & & -\frac{1}{2}\partial_{11} & \frac{1}{2}\partial_{12} & \frac{1}{2}\partial_{13} \\ & & & & -\frac{1}{2}\partial_{22} & \frac{1}{2}\partial_{23} \\ & & & & & -\frac{1}{2}\partial_{33} \end{bmatrix} \tag{29}$$

Substituting Eqs. (12) into Eqs. (27), and comparing it with Eqs. (26), we obtain

$$\Delta_i^* \nabla_t = \nabla_i^* \left(\frac{1}{\lambda_i} \nabla_t + \frac{1}{d_i} \right), \qquad i = 1,2,\cdots,6 \tag{30}$$

Equation (30) shows that there exists Maxwell's law between the stress operator and the strain operator under the condition of viscoelasticity. When the viscous coefficients $d_i \to \infty$ ($i = 1,2,\cdots,6$), it goes back to Hooke's law[8].

It is proved[8] that there exist the following potential functions for statics.

$$\dot{\sigma}_i^* = \nabla_i^* \nabla_t \psi_i, \qquad i = 1,2,\cdots,6 \tag{31}$$

$$\dot{\varepsilon}_i^* = \Delta_i^* \nabla_t \psi_i, \qquad i = 1,2,\cdots,6 \tag{32}$$

They contain both the equilibrium equation and the compatibility equation. By using Eqs. (26) and (27), the following equations are obtained

$$\Box_i^* \nabla_t \psi_i = 0, \qquad i = 1,2,\cdots,6 \tag{33}$$

where $\Box_i^* = \Delta_i^* \nabla_i^* = \nabla_i^* \Delta_i^*$, called the strain energy operator. So, Eqs. (26) and Eqs. (27) are indistinguishable for statics. No matter how complicated the anisotropy of solids may be, the fundamental equation of viscoelastic statics can be expressed with Eqs. (33).

Using Eqs. (30), Eqs. (33) becomes

$$\nabla_i^{*2} \left(\frac{1}{\lambda_i} \nabla_t + \frac{1}{d_i} \right) \psi_i = 0, \qquad i = 1,2,\cdots,6 \tag{34}$$

They are the definite equations used to solve the problem of anisotropic viscoelastic statics, and the corresponding modal boundary conditions are as follows[10]

$$\begin{aligned} \sum_i \nabla_i^* \psi_i (\varphi_{i1} l + \varphi_{i6} m + \varphi_{i5} n) &= X_N \\ \sum_i \nabla_i^* \psi_i (\varphi_{i6} l + \varphi_{i2} m + \varphi_{i4} n) &= Y_N \\ \sum_i \nabla_i^* \psi_i (\varphi_{i5} l + \varphi_{i4} m + \varphi_{i3} n) &= Z_N \end{aligned} \tag{35}$$

where φ_{ij} is jth element of ith modal vector.

It is necessary to show that the theory given here is based on Newton's viscous law, it can not be applied directly to the complicated viscoelastic solids, such as the constitutive equation with time differential term of higher order. But if regarding it as the basic solution, we can obtain the approximate one of the complicated viscoelastic mechanics with the little disturbing method[15].

V. APPLICATION

It does not lose generality that let us consider a plane problem of isotropic viscoelastic solids. Its structure of the mechanical space is ·

$$W = W_1^{(1)}(\boldsymbol{\varphi}_1) \oplus W_2^{(2)}(\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3) \tag{36}$$

Equation (36) shows that there are two subspaces in plane isotropic solids, in which eigenvector and eigen elasticity are respectively

$$\boldsymbol{\varphi}_1 = \frac{\sqrt{2}}{2}[1,1,0]^{\mathrm{T}}, \quad \boldsymbol{\varphi}_2 = \frac{\sqrt{2}}{2}[1,-1,0]^{\mathrm{T}}, \quad \boldsymbol{\varphi}_3 = [0,0,1]^{\mathrm{T}} \tag{37}$$

$$\lambda_1 = K, \quad \lambda_2 = G \tag{38}$$

where K and G are the volume modulus and shear modulus, respectively. So, the first subspace indicates the space of volume change and the second indicates the space of shear deformation for plane isotropic solids. The corresponding strain operators are calculated as follows:

$$\nabla_1^* = \frac{\sqrt{2}}{2}(\nabla_{11} + \nabla_{22}) \tag{39}$$

$$\nabla_2^* = \sqrt{\frac{(\nabla_{11} - \nabla_{22})^2}{2} + \nabla_{12}^2} \tag{40}$$

Because differentiation has nothing to do with sequence, that is, $\nabla_{11}\nabla_{22} = \nabla_{12}^2$, we have

$$\nabla_1^* = \nabla_2^* = \nabla^2 \tag{41}$$

where ∇^2 is Laplace's operator.

Thus, the fundamental equations of the plane problem of anisotropic viscoelastic solids becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \left(\frac{1}{K}\frac{\partial}{\partial t} + \frac{1}{d_v}\right)\psi_v = 0 \tag{42}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \left(\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{d_s}\right)\psi_s = 0 \tag{43}$$

In this case, the potential functions degenerate to Airy function, but the volume quantity equation and the shear one can not be united because of the existence of viscosity. When the volume viscous coefficients d_v and shear one d_s both tend to infinity, Eqs. (42) and (43) can go back to the double compatibility equation.

It is seen that the general solution of Eqs. (42) and (43) can not be obtained in the form of finite term, but the inverse or semi-inverse method which is similar to one of classical elastic mechanics can be used.

For example, let

$$\psi = -\tau_0 xyH(t) \tag{44}$$

where $H(t)$ is a step function.

It is obvious that Eq. (44) satisfies the fundamental equation, and we have

$$\begin{aligned} \sigma_1^* &= \sigma_x + \sigma_y = 0 \\ \sigma_2^* &= \tau_{xy} = \tau_0 H(t) \end{aligned} \tag{45}$$

Equations (45) describes the plane shear acting.

Substituting Eqs. (45) into Eqs. (12), and using $\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \delta(t)$, we obtain

$$\frac{\mathrm{d}\varepsilon_2^*}{\mathrm{d}t} = \frac{\tau_0}{G}\delta(t) + \frac{\tau_0}{d_s}H(t) \tag{46}$$

Integrating the above equation with time, and using $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$, and $\int_{-\infty}^{\infty} H(t)\mathrm{d}t = t$, we obtain

$$\varepsilon_2^* = \gamma_{xy} = \frac{\tau_0}{G} + \frac{\tau_0}{d_s}t \tag{47}$$

Equation (47) gives description of the elastic deformation and stable creep of Maxwell's solids.

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